A magic square

A mathematical vignette

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According to legend, a turtle climbed out of a ancient Chinese river. In its shell was inscribed a 3×3 square array of numbers, the digits from 1 to 9 inclusive, each appearing once is such a way that the three numbers in each row, in each column and in each of the two diagonals, all had the same sum. This is the original "magic square", and it is a nice exercise for pupils to reconstruct it.

Many pupils will intuit that 5 ought to go in the middle on the spurious grounds that 5 is the middle number among the nine digits. Generally, most will initially proceed through a process of trial and error. However, the magic square is a good place to illustrate the power of mathematics as a process of systematic analysis, and when students get into the proper mindset, they can come up with many ways of getting the answer.

Another issue that comes up is what it means for two magic squares to be essentially different, or in other words, to recognize that all the examples are equivalent in the sense that they can be related by rotations and reflections of the array.

Let us find out what the common sum along the rows, columns and diagonals must be. Since all the digits add up to 45 and there are three rows, the sum along each row must be 15. A good next step is to see how you can add three digits together and get 15. The possibilities are

$$15 = 9 + 5 + 1 = 9 + 4 + 2 = 8 + 6 + 1 = 8 + 5 + 2 = 8 + 4 + 3 = 7 + 6 + 2 = 7 + 5 + 3 = 6 + 5 + 4.$$

There are exactly eight possibilities. We have to accomodate eight sums in the square array, since there are three rows, three columns and two diagonals. So each of the eight sums is realized somewhere in the diagram. The number that goes in the middle cell must figure in four of the sums. Each corner digit is in three of the sums. and each mid-edge digit in two.

Up to a rotation and reflection of the figure, there is one answer:

4	3	8
9	5	1
2	7	6

There are a few more magic characteristics of this array. The products of the numbers in the three rows are 96, 45 and 84; added together, they give 225 which is the square of 15, the sum of each row. Similarly, the products of the numbers in the three columns are 72, 105 and 48; their sum is again the number 225.

This has a very nice generalization. Define a sequence $\{x_n\}$ for $n \geq 1$ by picking the first two entries arbitrarily and defining

$$x_{n+2} = ax_{n+1} + bx_n$$

for $n \geq 1$ where a, b are arbitrary constants. Thus, $\{x_n\}$ is a second order recursion. Replace each digit k by x_k. Then once again the sum of the row products is equal to the sum of the column products. If the students know the formula for the general term of a second order recursion, the proof is straightforward, albeit a little tedious.

There is more. Using the digits in the rows to create three three-digit numbers gives (438, 951, 276). Writing these numbers backwards gives (834, 159, 672). Not only two triples of three numbers have the same sum, their squares also add to the same thing:

$$438^2 + 951^2 + 276^2 = 834^2 + 159^2 + 672^2.$$

We can do a similar thing with the columns and find that the two triples (492, 357, 816) and (294, 753, 618) have the same sum and square-sum. If we go in the diagonal directions, we can two more pairs of triples with the same property:

$$(456, 231, 978), (654, 132, 879)$$

and

$$(852, 174, 639), (258, 471, 936).$$

We conclude with a paper-and-pencil game played by two people moving alternately. The first player picks a digit between 1 and 9. After that, each player picks a digit not already selected so far. The winner is the first person to find among the numbers she has selected, *three* that add up to 15. These three numbers need not be consecutive choices by the player. The game ends in a draw when all nine digits have been chosen and neither player has three summing to 15. Try this game out with a friend.

This game is isomorphic to noughts-and-crosses (tictactoe). To relate the two, have a magic square on hand, and when a player picks a number k, let him put his mark in the cell containing k. This plays off the fact that three numbers add to 15 if and only if they are in the same row, column or diagonal of a magic square. This is the nicest example that I know of to convey to the students what it means for two mathematical objects to have the same structure, while appearing to differ markedly.